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Classical and Quantum Integrability of 2D Dilaton Gravities in Euclidean space

L. Bergamin*, D. Grumiller†, W. Kummer‡ and D.V. Vassilevich§

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*†‡Institute for Theoretical Physics, TU Vienna,
Wiedner Hauptstr. 8–10/136, A-1040 Vienna, Austria

†§Institute for Theoretical Physics, University of Leipzig
Augustusplatz 10–11, D-04103 Leipzig, Germany

§V.A. Fock Institute of Physics, St. Petersburg University, Russia

Abstract

Euclidean dilaton gravity in two dimensions is studied exploiting its representation as a complexified first order gravity model. All local classical solutions are obtained. A global discussion reveals that for a given model only a restricted class of topologies is consistent with the metric and the dilaton. A particular case of string motivated Liouville gravity is studied in detail. Path integral quantisation in generic Euclidean dilaton gravity is performed non-perturbatively by analogy to the Minkowskian case.

*bergamin@tph.tuwien.ac.at
†grumiller@itp.uni-leipzig.de
‡wkummer@tph.tuwien.ac.at
§Dmitri.Vassilevich@itp.uni-leipzig.de

1 Introduction

Two-dimensional gravities attracted much attention over the last decades since they retain many properties of their higher-dimensional counterparts but are considerably more simple. It has been demonstrated that all two-dimensional dilaton gravities are not only classically integrable, but this even holds at the (non-perturbative) quantum level (for a recent review cf. ref. [1]). However, these integrability statements have been obtained for the Minkowski signature of the space-time only. Many applications require Euclidean signature as well. The most prominent example is string theory where the genus expansion (a typical Euclidean notion) plays an important role. Spectacular achievements in the Liouville model [2], where higher order correlation functions have been calculated and relations to matrix models have been established, are the main motivation for our present work.

The prime goal of this paper is to show classical and quantum integrability of two-dimensional dilaton gravities in Euclidean space. Our main example will be (generalised) Liouville gravity, which will be studied in full detail locally and globally.

We start with classical solutions. For a generic model all local solutions are obtained exploiting the complexified first order formalism and some general statements are made regarding the global structure (sect. 2). Then we turn to Liouville gravity, which we re-derive from the bosonic string sigma model (sect. 3). A peculiar feature of our approach is that we do not fix the conformal gauge and keep all components of the metric dynamical, as well as the Liouville field (which we call the dilaton, to keep the terminology consistent with other 2D models). For a three-parameter family of Liouville type gravities we construct all local and global solutions (sect. 4). Quantisation of Euclidean models (sect. 5) appears to be somewhat more complicated than the one of their Minkowski signature counterparts (which can be traced to the absence of the light-cone condition in the Euclidean space). Nevertheless, an exact non-perturbative path integral is performed and local quantum triviality is demonstrated. However, this does not mean that all correlators are trivial. As an illustration, we calculate some non-local correlation functions. In sect. 6 we briefly discuss perspectives of our approach. Appendix A recalls the equivalence between second and first order formulations for Euclidean signature while appendix B considers finite volume corrections to the Liouville action.

2 Classical solutions

2.1 The model

The action of general dilaton gravity [3, 4] (cf. also [5–7])

$$L_{\text{dil}} = \int d^2\sigma\sqrt{g}\left[-R\frac{\Phi}{2} + (\nabla\Phi)^2\frac{U(\Phi)}{2} + V(\Phi)\,,\right] \tag{1}$$

contains two arbitrary functions $U(\Phi)$ and $V(\Phi)$. An eventual dependence on an additional function $Z(\Phi)$ in the first term can be eliminated by a field redefinition if Z is invertible.[1]

As for Minkowskian signature it is convenient to work with the equivalent first order action [9, 10]

$$L = \int_{\mathcal{M}} \left[Y^a De^a + \Phi d\omega + \epsilon\mathcal{V}(Y^2, \Phi)\right]\,, \tag{2}$$

where $Y^2 = Y^aY^a = (Y^1)^2 + (Y^2)^2$, e^a is the Zweibein 1-form, the 1-form ω is related to the spin-connection by means of $\omega^{ab} = \varepsilon^{ab}\omega$, with the total antisymmetric ε-symbol, and the torsion 2-form reads explicitly

$$De^a = de^a + \omega^{ab}\wedge e^b\,. \tag{3}$$

The volume 2-form is denoted by $\epsilon = -\frac{1}{2}\varepsilon^{ab}e^a\wedge e^b$. One may raise and lower the flat indices $a, b, \dots$ with the flat Euclidean metric $\delta_{ab} = \text{diag}(1, 1)$, so there is no essential difference between objects with upper indices and objects with lower ones. Therefore, we do not discriminate between them. For a more detailed explanation of our notation and conventions of signs see appendix A. In this paper we mostly use the Cartan formalism of differential forms. An introduction to this formalism can be found in [11, 12], specific two-dimensional features are explained in [1]. Different models are encoded by the “potential” $\mathcal{V}$. Here, we restrict ourselves to

$$\mathcal{V} = \frac{1}{2}Y^2U(\Phi) + V(\Phi)\,, \tag{4}$$

although our analysis can be extended easily to a more general choices. The action (2) with the potential (4) describes practically all gravity models in two dimensions, e.g. the Witten black hole ($U = -1/(2\Phi)$, $V = -\lambda^2\Phi$) and the Schwarzschild black hole ($U = -1/(2\Phi)$, $V = -\lambda^2$). Important exceptions are dilaton-shift invariant models [13] inspired by the so-called exact string black hole [14], an action of which has been constructed only recently [15].

Equivalence of second and first order formalisms (with the choice (4)) is proved in appendix A, where it is also explained how to translate our notations to the component language.

[1] If it is not invertible generically curvature singularities arise for $Z' = 0$. Thus, by demanding invertibility we merely exclude cases with further singularities leading to global properties which do not show generic differences with respect to those to be encoded in $U(\Phi)$ and $V(\Phi)$ [8].

2.2 Local analysis

In terms of complex fields

$$Y = \frac{1}{\sqrt{2}}\left(Y^1 + iY^2\right) , \qquad \bar{Y} = \frac{1}{\sqrt{2}}\left(Y^1 - iY^2\right) ,$$
$$e = \frac{1}{\sqrt{2}}\left(e^1 + ie^2\right) , \qquad \bar{e} = \frac{1}{\sqrt{2}}\left(e^1 - ie^2\right) , \tag{5}$$

the action (2) reads

$$L = \int_{\mathcal{M}} \left[\bar{Y} De + Y\overline{De} + \Phi d\omega + \epsilon \mathcal{V}(2\bar{Y}Y, \Phi)\right] , \tag{6}$$

where

$$De := de - i\omega \wedge e \,, \qquad \overline{De} := d\bar{e} + i\omega \wedge \bar{e} \,. \tag{7}$$

The volume 2-form is defined as

$$\epsilon := i\bar{e} \wedge e \,. \tag{8}$$

We at first disregard the conditions implied by (5) and solve the model for arbitrary complex fields $\bar{Y}$, Y, $\bar{e}$ and e. If we also allow for complex ω, we may treat Euclidean and Minkowskian theories simultaneously. In Minkowski space $\{Y, \bar{Y}, e, \bar{e}, \omega\}$ should be replaced by $\{Y^+, Y^-, ie^+, ie^-, i\omega\}$ where all fields $Y^\pm$, $e^\pm$, ω are real (cf. [1]). The superscript $\pm$ denotes the light-cone components. Euclidean signature imposes a different set of reality conditions to be derived below.

Our complex model shares many similarities with complex Ashtekar gravity in four dimensions [16] and relates with the so-called generalised Wick transformation [17]. A two-dimensional version (spherical reduction) has been discussed in [18]. The main difference from usual theories of complex fields (say, from $|\phi|^4$ theory) is that if no relation between $\bar{Y}$, Y, $\bar{e}$ and e is assumed the action (6) is holomorphic rather than real. Path integration in such theories requires the contour integration in the complex plane rather than the Gaussian integration [19].

Another reason to study complex gravity theories is their relation to noncommutative models. In that case the Lorentz group does not close and one has to consider a complexified version [20, 21]. In two dimensions, a noncommutative version of Jackiw-Teitelboim gravity [22–25] was constructed in [26] and later quantised in [27]. Gauged noncommutative Wess-Zumino-Witten models have been studied in ref. [28]. Complexified Liouville gravity was shown to be equivalent to an $SL(2, R)/U(1)$ model [29].

The equations of motion from variation of ω, $\bar{e}$, e, Φ, $\bar{Y}$ and Y, respectively,

read

$$
\begin{aligned}
&d\Phi - i\bar{Y}e + iY\bar{e} = 0\,, && (9)\\
&DY + ie\mathcal{V} = 0\,, && (10)\\
&\overline{DY} - i\bar{e}\mathcal{V} = 0\,, && (11)\\
&d\omega + \epsilon\frac{\partial\mathcal{V}}{\partial\Phi} = 0\,, && (12)\\
&De + \epsilon\frac{\partial\mathcal{V}}{\partial\bar{Y}} = 0\,, && (13)\\
&\overline{De} + \epsilon\frac{\partial\mathcal{V}}{\partial Y} = 0\,. && (14)
\end{aligned}
$$

To obtain the solution of (9)-(14) we follow the method of [1] (cf. [30] and for a different approach [31]). By using (9) - (11) it is easy to demonstrate that

$$
d(\bar{Y}Y) + \mathcal{V}d\Phi = 0\,. \tag{15}
$$

Noting that $Y^2/2 = \bar{Y}Y$ with the choice (4) for $\mathcal{V}$ one can integrate (15) to obtain the local conservation law

$$
d\mathcal{C} = 0, \qquad \mathcal{C} = w(\Phi) + e^{Q(\Phi)}\bar{Y}Y \tag{16}
$$

with

$$
Q(\Phi) = \int^{\Phi} U(y)dy\,, \qquad w(\Phi) = \int^{\Phi} e^{Q(y)}V(y)dy\,. \tag{17}
$$

Next we use (10) to express

$$
\omega = Z\mathcal{V} - i\frac{dY}{Y}\,, \tag{18}
$$

where

$$
Z := \frac{1}{Y}e\,. \tag{19}
$$

We have to assume[2] that $Y \neq 0$. Now, by means of (9) we can show that

$$
\bar{e} = i\frac{d\Phi}{Y} + Z\bar{Y}\,, \tag{20}
$$

and with (13) that

$$
dZ = d\Phi \wedge ZU(\Phi)\,. \tag{21}
$$

Equation (21) yields

$$
d\hat{Z} = 0\,, \qquad \hat{Z} := e^{-Q}Z\,. \tag{22}
$$

[2]For Euclidean signature $Y = 0$ implies $\bar{Y} = 0$. Solutions where $Y = 0 = \bar{Y}$ globally may be analysed separately and admit only constant curvature. Y and $\bar{Y}$ cannot vanish at isolated points inside the manifold.

This equation can be integrated locally[3]:

$$\hat{Z} = df \,, \tag{23}$$

where f is a complex valued zero-form. Besides (16) this is the only integration needed to obtain the solution of the model, which depends on two arbitrary complex functions, f and Y, one arbitrary real function Φ, and one constant of motion $\mathcal{C}$. $\bar{Y}$ is then defined by the conservation law (16), and the geometry is determined by

$$e = Y e^Q df \,, \tag{24}$$

$$\bar{e} = i\frac{d\Phi}{Y} + \bar{Y} e^Q df \,, \tag{25}$$

$$\omega = \mathcal{V} e^Q df - i\frac{dY}{Y} \,. \tag{26}$$

One can check easily that indeed all the equations (9) - (14) are satisfied. This completes the local classical analysis of complex dilaton gravity in 2D.

To proceed with real Euclidean dilaton gravity the two reality conditions are needed which follow from (5):

$$\bar{Y} = Y^* \,, \tag{27}$$

$$\bar{e} = e^* \,. \tag{28}$$

Here star denotes complex conjugation. The first of these conditions implies that $\mathcal{C}$ is real. The second one fixes the imaginary part of Z:

$$\Im Z = -\frac{1}{2}\frac{d\Phi}{\bar{Y}Y} \,. \tag{29}$$

The line element

$$(ds)^2 = 2\bar{e}e = 2\bar{Y}Y e^{2Q}(d\theta)^2 + \frac{(d\Phi)^2}{2\bar{Y}Y} \tag{30}$$

(with $\theta := \Re f$) is now explicitly real. The product $\bar{Y}Y$ can be eliminated by means of the conservation law (16) so that

$$(ds)^2 = e^{Q(\Phi)}\left(\xi(\Phi)(d\theta)^2 + \xi(\Phi)^{-1}(d\Phi)^2\right) \,, \quad \xi = 2(\mathcal{C} - w(\Phi)) \,. \tag{31}$$

This solution always has a Killing vector ∂_θ and depends on one constant of motion $\mathcal{C}$. The Levi-Civita connection $\hat{\omega}$ and the scalar curvature corresponding to the line element (31) read

$$\hat{\omega} = \frac{1}{2}(U\xi + \xi')d\theta \,, \qquad R = -e^{-Q}(U'\xi + U\xi' + \xi'') \,, \tag{32}$$

[3]Globally (23) is true up to a harmonic one-form (see sec. 2.3).

where prime denotes differentiation w.r.t. Φ.

It should be noted that whereas (24) and (25) would lead to an Eddington-Finkelstein type line element in complex gravity, reminiscent of the one in Minkowski space [1], the reality conditions (27) and (28) force a gauge of diagonal type. This is the source of many technical complications for Euclidean signature.

2.3 Global structure

Our strategy is borrowed from Lorentzian signature gravity theories. We take local solutions (31) and extend them as far as possible along geodesics. In this way we obtain all global solutions. Here we only sketch the methods, postponing a more detailed discussion to the next section where we treat Liouville gravity as a particular example.

In the Φ direction each solution can be extended as long as the metric is positive definite, $\xi > 0$, and as long as it does not hit a curvature singularity. Note that Φ plays a dual role. It is not only one of the coordinates, it is also a scalar field. Therefore, one cannot identify regions with different values of Φ as it would lead to a discontinuity. In the θ direction one can either extend the solution to an infinite interval or impose periodicity conditions. It is easy to see that a smooth solution can be obtained only if the period does not depend on Φ. This period should be chosen such that conical singularities are avoided at the roots of $\xi(\Phi)$. In some cases one can also "compactify" the manifold by adding a point with $\Phi = \infty$ (see below).

From a different point of view the action (2) can be considered as a particular case of a Poisson Sigma Model (PSM) [10, 32]. Classical solutions of PSMs on Riemann surfaces have been analysed recently in [33]. It has been found that on an arbitrary surface one encounters a $(1 + \dim H^1(\mathcal{M}))$-parametric family of solutions. Roughly speaking, one of the parameters is the conserved quantity $\mathcal{C}$. The others originate from arbitrariness in global solutions of (22) which is exactly the dimension of the first cohomology group of the underlying manifold $\mathcal{M}$.

Here we use a different approach. In a gravity theory, the topology of the manifold has to be consistent with the metric.[4] For example, solutions with $\xi < 0$ are perfectly admissible for a PSM, but make no sense in a Euclidean gravity theory. Also, curvature singularities play an important role in analysing the global structure of classical solutions in gravity, but are not discussed in the PSM approach [33].

In the particular case of the Katanaev–Volovich model [34, 35] a complete analysis of local and global solutions in the conformal gauge has been performed in ref. [36].

[4] An example of restrictions imposed on topology by the Riemannian structure is the Gauss-Bonnet density, the integral of which essentially provides the Euler characteristic.

3 Liouville gravity from bosonic strings

Before discussing in detail classical solutions of the Liouville gravity model, we re-derive the Liouville action from string theory. Our derivation does not imply conformal gauge fixing and, therefore, differs from more standard ones (cf. the discussion at the end of this section).

The starting point is the bosonic string partition function

$$\mathcal{Z}(l) = \int [\mathcal{D}g]\, \mathcal{D}X \exp\left(-L(X,g) - \mu_0 \int d^2\sigma \sqrt{g} + \int d^2\sigma l_A X^A \right) , \tag{33}$$

where the "matter part" corresponds to the string action

$$L(X,g) = \frac{1}{2} \int d^2\sigma \sqrt{g} g^{\mu\nu} \partial_\mu X^A \partial_\nu X_A \,. \tag{34}$$

Here X is a scalar field with values in $\mathbb{R}^N$. We have introduced sources l_A for X^A, which are scalar densities from the point of view of the world sheet. We assume Euclidean signature on the target space and on the world sheet. The measure $[\mathcal{D}g]$ includes all usual gauge parts (ghosts and gauge fixing).

One can integrate over X by using a procedure suggested by Polyakov [37] (cf. also an earlier paper [38]). The effective action $W(g,l)$ is defined by

$$\begin{aligned} W(g,l) \quad &= -\ln \int \mathcal{D}X\, e^{-L(X,g) + \int d^2\sigma l_A X^A} + \mu_0 \int d^2\sigma \sqrt{g} \\ &= \frac{N}{2} \ln\det(-\Delta) + \frac{1}{2} \int d^2\sigma \sqrt{g} l \Delta^{-1} l + \mu_0 \int d^2\sigma \sqrt{g} \,, \end{aligned} \tag{35}$$

where $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$. The presence of l allows us to calculate correlation functions of X although this field has been integrated out already. However, from now on we put $l = 0$ and define $W(g) = W(g,0)$.

We wish to retain one out of the N contributions in the first term on the right hand side of (35). The remaining $N-1$ terms can be fixed by the conformal anomaly

$$T_\mu^\mu = \frac{N-1}{24\pi} [R + \tilde{\mu}] \,, \tag{36}$$

where $T_{\mu\nu}$ is the vacuum expectation value of the energy-momentum tensor for $N-1$ fields X. R is the Riemann curvature of the two-dimensional metric $g_{\mu\nu}$. The term with $\tilde{\mu}$ describes renormalisation of the cosmological constant. It is "non-universal", i.e. it does not appear in certain regularisations, as, e.g., in the zeta function one.

Using the well-known relation between T_μ^μ and the effective action in the terms $\propto (N-1)$ yields[5]

$$W(g) = \frac{N-1}{96\pi} \int d^2\sigma \sqrt{g} \left[R \frac{1}{\Delta} R + \mu \right] + \frac{1}{2} \ln\det(-\Delta) \,, \tag{37}$$

[5] On a compact manifold the operator Δ has zero modes. Therefore, the argument has to be modified (cf. Appendix B).

where the constant μ contains contributions from μ_0 and $\tilde{\mu}$.

This action can be transformed to a local one by introducing an additional scalar field Φ,

$$W(g,\Phi) = \frac{N-1}{24\pi} \int d^2\sigma \sqrt{g} \left[(\partial\Phi)^2 - \Phi R + \mu/4 \right] , \tag{38}$$

so that the partition function (33) becomes

$$\mathcal{Z} = \int [\mathcal{D}g] \, \mathcal{D}\Phi e^{-W(g,\Phi)} . \tag{39}$$

The integration over Φ produces both the non-local term and the $\ln\det(-\Delta)$ in (37) which had been separated there just for this purpose. One could perform a rescaling $\Phi \to \sqrt{N-1}\Phi$. Then the limit $N \to 1$ is consistent (but not particularly interesting) as it reproduces our starting point (33) without external sources.

An exponential interaction term can be obtained by a conformal transformation of the metric:

$$W(e^{\alpha\Phi}\tilde{g},\Phi) = \frac{N-1}{24\pi} \int d^2\sigma \sqrt{\tilde{g}} \left[(1-\alpha)(\partial\Phi)^2 - \Phi\tilde{R} + \mu e^{\alpha\Phi}/4 \right] , \tag{40}$$

where α is a constant. Note that the conformal transformation $g \to \tilde{g}$ is singular for $\Phi \to \pm\infty$.

It is instructive to compare our approach to the one of Ref. [39,40], which also derives the Liouville action from quantum strings. There this action is obtained from fixing the conformal gauge while we integrate over the string coordinates X^A. As a result, the action of Refs. [39,40] contains somewhat different numerical coefficients, and, more important, the metric there is non-dynamical. If one keeps the sources l^A for X^A, both approaches should give equivalent quantum theories. As we shall see below, our results for geometries of "Liouville gravity" are consistent with the semiclassical results in the conformal gauge approach (cf. e.g. [41]).

4 Classical solutions of Liouville gravity

In accordance with (40) we restrict ourselves to the first order action (2) with a potential of type (4) where

$$U(\Phi) = a \, , \qquad V(\Phi) = b e^{\alpha\Phi} \, , \tag{41}$$

with $a, b, \alpha \in \mathbb{R}$. Note that one of the constants can be absorbed in a redefinition of Φ. From (17) one immediately finds that for $(a+\alpha) \neq 0$

$$Q(\Phi) = a\Phi, \qquad w(\Phi) = \frac{b}{a+\alpha} e^{(a+\alpha)\Phi} . \tag{42}$$

The simple case $a+\alpha = 0$ yields $w = b\Phi$ and will be treated separately. Equation (31) becomes

$$(ds)^2 = e^{a\Phi}\left(\xi(\Phi)(d\theta)^2 + \xi(\Phi)^{-1}(d\Phi)^2\right),$$
$$\xi = 2\left(\mathcal{C} - \frac{b}{a+\alpha}e^{(a+\alpha)\Phi}\right), \tag{43}$$

yielding the scalar curvature:

$$R = 2be^{\alpha\Phi}(2a+\alpha)\,. \tag{44}$$

It should be emphasised that R is independent of $\mathcal{C}$, a feature which is *not* true generically but holds if and only if $U = \text{const.}$, as seen from (32). At $\alpha\Phi \to -\infty$ we have an asymptotically flat region; $\alpha\Phi \to +\infty$ corresponds to a curvature singularity. At $\Phi = \Phi_0$ such that $\xi(\Phi_0) = 0$ there is a coordinate singularity. In a space with Minkowski signature $\Phi = \Phi_0$ corresponds to a Killing horizon.

Depending on the sign of a, b and α different global geometries are possible. We consider them case by case.

4.1 Generic solutions

First we assume $a > 0$. The case $a < 0$ can be obtained by changing the sign of Φ. The limit $a = 0$ is addressed in the subsequent subsection.

Case I: $b, \alpha > 0$.

For $\mathcal{C} \le 0$ the metric (43) is negative definite, so that no solution in Euclidean space exists. For $\mathcal{C} > 0$, the coordinate Φ is restricted by the inequality

$$\Phi \le \Phi_0 = \frac{1}{a+\alpha}\ln\left(\frac{(a+\alpha)\mathcal{C}}{b}\right), \tag{45}$$

so that the solution includes the asymptotically flat region but does not contain any curvature singularity.

Near Φ_0 the line element (43) becomes

$$(ds)^2 \approx e^{a\Phi_0}\left(2(a+\alpha)\mathcal{C}\phi(d\theta)^2 + \frac{(d\phi)^2}{2(a+\alpha)\mathcal{C}\phi}\right), \tag{46}$$

where $\phi := \Phi - \Phi_0$. To avoid a conical singularity at $\phi = 0$ the coordinate θ should be periodic with the period

$$\theta_0 = \frac{2\pi}{(a+\alpha)\mathcal{C}}\,. \tag{47}$$

We see that these solutions correspond to a characteristic temperature which, however, cannot be identified directly with $1/\theta_0$ since the metric (43) is not explicitly the unit one in the flat region $\Phi \to -\infty$.

Although the coordinate Φ varies from $-\infty$ to Φ_0, the proper geodesic distance along the line $\theta = const.$ between $\Phi = -\infty$ and $\Phi = \Phi_0$ is finite (and easy to calculate but looks ugly). This suggests that $\Phi = -\infty$ is a boundary, and we have a manifold with the topology of a disc. To make sure that this is really the case, let us calculate the Euler characteristic of the manifold. The volume contribution

$$\chi_{\mathrm{vol}} = \frac{1}{4\pi} \int_{\mathcal{M}} R\sqrt{g} d^2\sigma = \frac{2a+\alpha}{a+\alpha} \tag{48}$$

is not an integer. However, if we add the boundary part

$$\chi_{\mathrm{bou}} = \frac{1}{2\pi} \int_{\Phi=-\infty} k d\tau = -\frac{a}{a+\alpha} \tag{49}$$

(where k is the trace of the second fundamental form of the boundary, and $d\tau$ is the arc length along the boundary), we obtain

$$\chi = \chi_{\mathrm{vol}} + \chi_{\mathrm{bou}} = 1\,, \tag{50}$$

which coincides with the Euler characteristic of the disc.

This metric exhibits a somewhat strange behaviour: the radius of the circle $\Phi = const$ goes to zero as $\Phi \to -\infty$. Therefore, it seems natural to add that point to the manifold. This has to be done smoothly. To avoid a conical singularity there the period of the θ coordinate should be

$$\tilde{\theta}_0 = \frac{2\pi}{\mathcal{C}a}\,. \tag{51}$$

A smooth manifold can only be achieved if $\tilde{\theta}_0 = \theta_0$ yielding $\alpha = 0$. This special case is considered below (Case sI).

Adding a point to the manifold is only possible at $\Phi = \pm\infty$. The point we add should be at a finite distance from the points inside the manifold, should correspond to a region with a finite curvature, and the circles $\Phi = const$ should have zero radius at this point. All these restrictions can be satisfied for the cases I (above) and sI (below).

Case II: $\alpha > 0$, $b < 0$.

Here both signs of the conserved quantity $\mathcal{C}$ are possible. For $\mathcal{C} < 0$ the coordinate Φ ranges from Φ_0 to $+\infty$ (curvature singularity). The conical singularity at $\Phi = \Phi_0$ is avoided by imposing the same periodicity condition (47) on θ. The geodesic distance between $\Phi = \Phi_0$ and $\Phi = +\infty$ is finite.

For $\mathcal{C} > 0$ there is no point such that $\xi(\Phi) = 0$. Therefore, the period in θ is not fixed and $\Phi \in [-\infty, \infty[$. The flat region ($\Phi = -\infty$) is separated by a finite distance from the curvature singularity ($\Phi = +\infty$). Since all solutions contain a singularity this case does not fit into the discussion of standard topologies.

Case III: $\alpha < 0$, $a + \alpha > 0$.

For $b > 0$ solutions with $\mathcal{C} < 0$ are excluded. If $\mathcal{C} > 0$, then $\Phi \in]-\infty, \Phi_0]$, i.e. all solutions contain a curvature singularity and have to be periodic. The distance to the singularity is finite.

For $b < 0$ and $\mathcal{C} < 0$ we encounter non-compact periodic solutions with $\Phi \in [\Phi_0, +\infty[$, which includes the flat region but not the singularity. The total distance along the geodesic $\theta = const$ is infinite. This is a deformed Euclidean plane. The situation for $\mathcal{C} > 0$ is similar, but then all solutions also include the curvature singularity.

Case IV: $a > 0$, $\alpha < 0$, $a + \alpha < 0$.

This case can be easily analysed along the same lines, exchanging $b \to -b$.

4.2 Special solutions

Case sI: $\alpha = 0$, $a, b \neq 0$

This is a rather special[6] but the most important particular case since it corresponds to the original action (38) before the conformal transformation. Obviously, the sign of a is not significant as it can be reabsorbed into a reflection of Φ and b. We therefore take $a > 0$. For $b > 0$ negative values of $\mathcal{C}$ are not allowed. For positive $\mathcal{C}$ we encounter a manifold with constant positive scalar curvature which can be identified with a two-sphere S^2 after adding a point corresponding to $\Phi = -\infty$. Adding this point removes the boundary contribution from (50) and for $\alpha = 0$ Eq. (48) establishes $\chi = 2$ as required for S^2. For $b < 0$ we have hyperbolic spaces with negative constant scalar curvature. For $\mathcal{C} > 0$ the solution[7] should be completed by $\Phi = +\infty$ and θ should be taken periodic with the period (51).

Case sII: $a = 0$, $\alpha, b \neq 0$

Taking the formal limit $a \to 0$ in (49) and (48) provides $\chi = 1$, i.e. the result for a disc. However, depending on the signs of α and b singularities may be encountered and the manifold need not be compact. In fact, the discussion is in full analogy to the one for generic solutions.

[6] It is evident from (44) that curvature is constant for $\alpha = 0$, just like for the Jackiw-Teitelboim model [22–25]. Nevertheless, the resulting action is inequivalent to the one of that model because both potentials (41) differ from $U_{\text{JT}} = 0$, $V_{\text{JT}} \propto \Phi$. It is not even conformally related because the conformally invariant function w in (42) is not quadratic in the dilaton Φ, as required for the Jackiw-Teitelboim model and its conformally related cousins.

[7] This case corresponds to the Katanaev-Volovich model [34, 35] for vanishing R^2 term, i.e., a theory with constant torsion and cosmological constant. Its Euclidean formulation has been discussed in ref. [42].

Case sIII: $a + \alpha = 0$, $b \neq 0$

Instead of the second equation in (42) one obtains $w = b\Phi$ and the curvature becomes $R = 2abe^{-a\Phi}$. Thus one arrives at a conformally transformed version of the Witten black hole [43–45] with topology of a "cigar". A conformal frame that is often employed [46] exploits the simplicity of the geometry for $a = 0$: the metric is flat.

Case sIV: $b = 0$

Here the value of α is irrelevant and the solutions are flat. The period reads $\theta_0 = 2\pi/(\mathcal{C}a)$. If additionally $a = 0$ both potentials vanish and toric topology is possible by identifying Φ periodically. Then the Euler characteristic vanishes trivially. This is the only case where the Euler characteristic is not positive.

4.3 Concluding remarks

It is a common feature of all finite volume solutions that they are periodic in θ and do not include the region $a\Phi \to +\infty$. This completes our analysis of classical solutions in Liouville gravity.

A final remark in this section regards the Euler characteristic which worked so well in the Case I above. To define χ for a non-compact manifold, it has to be placed into a box. Otherwise, the curvature integral yields an incorrect (infinite or non-integer) value which is not related to the index [11].

5 Quantisation

The quantisation for Minkowskian signature in the absence of matter has been performed for various dilaton gravity models in different formalisms. It is not necessary to review all these approaches and results (cf. [1] and references therein, especially sections 7.-9.; see also [47]). Here we follow the path integral quantisation [48, 49].

Employing the complex formalism of sect. 2.2 there are two possible strategies: either to quantise first and to impose the reality condition afterwards by hand or the other way round. Clearly, the second route is the one that *should* (and will be) be pursued[8]. Alas, at first glance it appears that it is only the first one that *can* be pursued. In the following, we will discuss why and how.

The crucial technical ingredient which makes all path integrations for Minkowskian signature straightforward was to impose a temporal gauge for light-like Cartan variables $e^{\pm}$, which corresponds to the Eddington-Finkelstein gauge in the metric [48, 49]. This is quite easy to understand: the action of type (2) is bilinear in the gauge fields ω and e^a. If e.g. the components ω_1, e_1^a are gauged

[8] It should be noted that the path integral quantisation of complex gravity [19] is based upon equivalence to a real version.

to constants then the gauge fixed action becomes linear in the remaining field components. Provided that no complications arise from the ghost sector (which indeed happens to be true) there will be as many functional δ-functions as there are gauge fields, which in turn can be used to integrate out the target space coordinates Φ, Y^a. As a result the quantum effective action is nothing but the *classical* action, up to boundary contributions.

5.1 Gauge fixing and path integral

The concept of the Eddington-Finkelstein gauge is intimately related to the Minkowskian signature, but it can be generalised readily to complex fields, e.g.

$$e_1 = 0\,, \quad \bar{e}_1 = c\,, \quad \omega_1 = 0\,, \tag{52}$$

with some non-vanishing $c \in \mathbb{C}$. Thus, the program as outlined above can be applied to quantise complex dilaton gravity in the first order formalism. Again, local quantum triviality can be established and the quantum effective action equals the classical one, up to boundary contributions. One can now try to impose the reality conditions (27), (28) on the mean fields appearing in that action to obtain Euclidean signature. However, on general grounds it is clear that imposing such conditions and quantisation need not commute; apart from that, the reality condition (28) is incompatible with the gauge choice (52) for reasons described below. Thus, if one is interested in rigorous quantisation of Euclidean dilaton gravity one ought to impose the reality conditions (27), (28) *first*. This leads to a serious problem, namely that the notion of "light-like" field components ceases to make sense and the gauge (52) no longer is applicable. The fastest way to check this statement is to recognise that (28) implies $c = 0$ in (52) which amounts to a singular gauge with degenerate metric. In other words, such a choice is not accessible for Euclidean reality conditions and has to be discarded.

These difficulties can be avoided if instead of (52) we employ

$$e_1 = c\,, \quad \bar{e}_1 = c^*\,, \quad \omega_1 = 0\,, \tag{53}$$

with some non-vanishing $c \in \mathbb{C}$. Such a gauge is compatible with the reality condition (28). The metric becomes

$$g_{\mu\nu} = \begin{pmatrix} 2|c|^2 & ce_2^* + c^* e_2 \\ ce_2^* + c^* e_2 & 2|e_2|^2 \end{pmatrix} . \tag{54}$$

The path integral quantisation based upon the formulation (2) now can be performed in full analogy to [49] (see also section 7 of [1]). As there arise no subtleties in the ghost and gauge fixing sector after (trivially) integrating out e_1, e_1^*, ω_1 and the ghosts we only obtain a contribution $\mathcal{F}$ to the Faddeev-Popov

determinant:

$$\mathcal{F} = \det \begin{pmatrix} -\partial_1 & -ic^*\mathcal{V}' & ic\mathcal{V}' \\ ic^* & -ic^*Y^*\dot{\mathcal{V}} & -\partial_1 + icY^*\dot{\mathcal{V}} \\ -ic & -\partial_1 - ic^*Y\dot{\mathcal{V}} & icY\dot{\mathcal{V}} \end{pmatrix} \tag{55}$$

Here $\mathcal{V}' = \partial_\Phi \mathcal{V}$ and $\dot{\mathcal{V}} = \partial_{YY^*}\mathcal{V}$. This result can be checked easily by using a gauge fixing fermion implying a "multiplier gauge" [50]. Then, the matrix $\mathcal{F}$ is given by the "structure functions" in full analogy to the Minkowskian case (cf. Eqs. (E.117) and (E.118) of [50]). The only difference is that instead of containing a single ghost-momentum the gauge fixing fermion now contains two ghost-momenta, multiplied by c, c^*, respectively. Consequently, also $\mathcal{F}$ consists of two terms in such a way that a real expression emerges. The crucial observation is that the determinant (55) is independent from ω and $e^\pm$.

After this step the generating functional for the Green functions reads

$$\begin{aligned} \mathcal{Z}(j, J) = &\int \mathcal{D}e_2 \mathcal{D}e_2^* \mathcal{D}\omega_2 \mathcal{D}Y \mathcal{D}Y^* \mathcal{D}\Phi\, \mathcal{F} \\ &\times \exp\left[i \int d^2x \left(L_{\text{g.f}} + je_2 + \bar{j}e_2^* + j_\omega \omega_2 + JY + \bar{J}Y^* + J_\Phi \Phi \right) \right], \end{aligned} \tag{56}$$

where j, J are external sources. Reality implies $j_\omega \in \mathbb{R}$ and $\bar{j} = j^*$. As a result, the classical action has been replaced by its gauge-fixed version

$$L_{\text{g.f.}} = Y^*(\partial_1 e_2 + ic\omega_2) + Y(\partial_1 e_2^* - ic^*\omega_2) + \Phi\partial_1\omega_2 + i(c^*e_2 - ce_2^*)\mathcal{V}\,. \tag{57}$$

Note that the reality conditions (27), (28) already have been imposed in (57). However, one can undo this restriction simply by replacing all quantities with $*$ by barred quantities.

Due to the linearity of (57) in e_2, e_2^* and ω_2, and due to the independence of (55) on these variables the path integration over those components produces three functional δ functions with arguments

$$-\partial_1\Phi + icY^* - ic^*Y + j_\omega\,, \tag{58}$$

$$-\partial_1 Y^* + ic^*\mathcal{V}(2YY^*, \Phi) + \bar{j}\,, \tag{59}$$

$$-\partial_1 Y - ic\mathcal{V}(2YY^*, \Phi) + j\,. \tag{60}$$

It is useful to introduce the real fields

$$Y_1 := c^*Y + cY^*\,, \quad Y_2 := i\left(cY^* - c^*Y\right)\,. \tag{61}$$

This redefinition yields a constant (nonsingular) Jacobian, as well as $2YY^* = (Y_1^2 + Y_2^2)/(2|c|^2)$. Integrating out the target space variables with (4) establishes

$$\partial_1\Phi = Y_2 + j_\omega\,, \tag{62}$$

$$\partial_1 Y_1 = j_1\,, \tag{63}$$

$$\partial_1 Y_2 = -\frac{1}{2}(Y_1^2 + Y_2^2)U(\Phi) - 2|c|^2 V(\Phi) + j_2\,, \tag{64}$$

where $j_1 := cj^* + c^*j$ and $j_2 := i(cj^* - c^*j)$. The inverse determinant which appears due to this integration contains the same combination of structure functions appearing in the matrix (55). Thus, as in the Minkowskian case, there is no Faddeev-Popov determinant left after integrating over *all* variables. For generic gauges an analogous statement can be found in ref. [51].

In the absence of sources eq. (63) can be integrated immediately, while eqs. (62) and (64) are coupled. Nevertheless, by taking a proper combination of them it is possible to reproduce the conservation equation (15), which then can be integrated as in (16). This equation can be used to express Y_2 in terms of Y_1 and of Φ. Then, only the first order equation (62) has to be solved. Including source terms[9] j_1, j_2 the general solution for Y_1, Y_2 can be represented as

$$Y_1 = Y_1^h + \partial_1^{-1} j_1\,, \qquad Y_1^h \in \mathbb{R}\,, \tag{65}$$

$$(Y_2)^2 = 4|c|^2 e^{-Q(\Phi)}\left(\mathcal{C}_0 - w(\Phi)\right) - (Y_1)^2 + 2\hat{T}\left(j_2 \partial_1 \Phi + Y_1 j_1\right)\,, \tag{66}$$

with

$$\hat{T}(f(x^1)) := e^{-Q(\Phi)} \partial_1^{-1}\left(e^{Q(\Phi)} f(x^1)\right)\,. \tag{67}$$

Inserting one of the square-roots of (66) into (62) yields a non-linear equation for Φ which reduces to a first order differential equation in the absence of sources. Note that the integration functions Y_1^h and $\mathcal{C}_0$ are not only independent from x^1 but show also independence[10] from x^2. In the absence of sources the Casimir function $\mathcal{C}$ coincides on-shell with the constant $\mathcal{C}_0$. But even when sources are present the quantity $\mathcal{C}_0$ remains a free constant and therefore fixing it to a certain value can be used to define the physical vacuum.

To summarise, the final path integration yields the generating functional for Green functions,

$$\mathcal{Z}(j, J) = \exp\left[i \int d^2x \left(J_\Phi \hat{\Phi}(j) + J^1 \hat{Y}_1(j) + J^2 \hat{Y}_2(j) + L_{\text{amb}}(j)\right)\right]\,, \tag{68}$$

where J^1, J^2 are appropriate linear combinations of $J, \bar{J}$ (cf. (61)) and $\hat{\Phi}$, $\hat{Y}_1$ and $\hat{Y}_2$ are the solutions obtained from (65)-(66) in the presence of sources and for a certain choice of the integration constants, in particular $\mathcal{C}_0$. Ambiguities which result from a careful definition of the first three terms in (68) are collected in

[9] For simplicity j_ω has been set to zero which is sufficient to describe situations where only the dependence on the metric is important. For $j_\omega \neq 0$ it is still possible to solve (65) (which remains unchanged) and (66), which receives additional contributions non-linear in j_ω. These terms have to be added on the r.h.s. of (66), but the modified version of (66) still is quadratic (and algebraic) in Y_2.

[10] It should be emphasised that in addition to (62)-(64) there are the so-called "lost constraints" [50,52], which play the role of gauge Ward identities and are nothing but the classical equations of motion analogous to (62)-(64) but with ∂_1 replaced by ∂_2. Moreover no additional integration constants arise from the equations of motion for the gauge fields if these "lost constraints" are taken into account [53].

$L_{\rm amb}(j)$. Their explicit form is not relevant for the current work. The curious reader may wish to consult either section 7 of [1] or some of the original literature [52, 54] where these terms are discussed in detail. We just recall a heuristic argument showing their necessity: it is seen clearly from (68) that their absence would imply

$$\langle e^a \rangle = \left. \frac{\delta \ln \mathcal{Z}}{\delta j_a} \right|_{j=J=0} = 0 = \left. \frac{\delta \ln \mathcal{Z}}{\delta j_\omega} \right|_{j=J=0} = \langle \omega \rangle \,, \tag{69}$$

because differentiation with respect to the sources j and setting $J = j = 0$ yields no contribution from the first three terms of the r.h.s. of (68). Thus, $L_{\rm amb}$ encodes the expectation values of connection and Zweibeine.

A final remark is in order: because the theory does not allow propagating physical modes the generating functional for Green functions (68) solely[11] depends on the external sources, and it does so in a very specific way: the sources J appear only linearly, while the dependence on j generically is non-polynomial. This has important consequences for clustering properties of correlators. For instance, expectation values of the form $\langle f(\Phi, Y_1, Y_2) \rangle$, where f is an arbitrary function of Φ, Y_1 and Y_2, are just given by their classical value $f(\Phi, Y_1, Y_2)$ ("the correlators cluster"), while expectation values containing at least one insertion of either Zweibein or connection generically receive non-trivial corrections in addition to the classical terms. This point will be considered in more detail below. A particular consequence for the Casimir (16) is

$$\langle \mathcal{C} \rangle = \mathcal{C}_0 \,, \tag{70}$$

where $\mathcal{C}_0$ is the integration constant appearing in (66). By choosing a particular value of $\mathcal{C}_0$ one fixes the vacuum state.

5.2 Solution in absence of sources

In the absence of sources the solutions of (62) - (64) simplify to

$$Y_1 = Y_1^h \,, \tag{71}$$

$$Y_2 = \pm \sqrt{4|c|^2 e^{-Q(\Phi)} \left(\mathcal{C} - w(\Phi)\right) - (Y_1^h)^2} \,, \tag{72}$$

$$\int^\Phi \frac{d\phi}{Y_2(\phi)} = x^1 - \bar{x}^1 \,, \tag{73}$$

where $\bar{x}_1 \in \mathbb{R}$ is the third integration constant. It can always be absorbed by a trivial shift of the coordinate x^1 and thus may be set to zero. For models

[11] Of course, as discussed before it also depends on various integration constants, most of which may be absorbed by trivial redefinitions – so essentially it depends on $\mathcal{C}_0$ besides the sources. However, this dependence is a parametric and not a functional one.

without torsion (Q = const.) or for Euclidean Ground state models (defined by $e^{-Q(\Phi)}w(\Phi)$ = const.) the integration constant Y_1^h may be absorbed into a redefinition of c (apart from certain singular values). In these cases only one essential integration constant remains, namely the Casimir $\mathcal{C}$. As a technical sidenote, depending on the specific choice of the potentials U, V (which then determine w, Q) the integration of (73) in terms of known functions may be possible. The Liouville model for $\alpha = 0$ will be treated below as a particular example.

The existence of two branches of the square root in (72) represents a delicate point. The crucial observation here is that, apart from diffeomorphisms, our gauge group actually is $O(2)$ which differs from $SO(2)$ by a discrete symmetry, namely reflections. Another way to see this fact is that the gauge orbits $Y_1^2 + Y_2^2$ = const. are circles for Euclidean signature (as opposed to hyperbolas for Minkowskian signature);[12] thus, fixing the value of Y_1 by choosing some Y_1^h and fixing $|Y_2|$ by (66) is not enough—one encounters two "Gribov copies". Suppose for simplicity $c \in \mathbb{R}$. Then, even after fixing the gauge according to (53) one may still perform a residual discrete gauge transformation, namely exchanging all barred with unbarred quantities, corresponding to complex conjugation. Because the action is real it is invariant, and the gauge (53) for real c is also invariant.[13] But $Y_2 \to -Y_2$ under this discrete transformation. Thus, fixing the sign in (72) removes this residual gauge freedom.

Nevertheless, in practice the best strategy is to keep both signs, to evaluate Φ for both and to choose whether one would like to have Φ monotonically increasing ($Y_2 > 0$) or decreasing ($Y_2 < 0$) as a function of x^1 at a certain point—e.g. in the asymptotic region, whenever this notion makes sense.

5.3 Example: Liouville model with $\alpha = 0$

For the Liouville model (41) with $\alpha = 0$ there exists an alternative way to solve the system (62)-(64). As (64) does not depend on Φ one can simply solve this equation without invoking the conservation law and plug the result directly into

[12] That is the reason why this issue never arose in the Minkowskian case. In Minkowskian space the solutions of Y^+Y^- = const. are hyperbolas and thus two branches exist. Fixing e.g. $Y^+ = +1$ determines not only Y^+ uniquely, but also the sign of Y^-! In contrast, for Euclidean signature the solutions $Y_1^2 + Y_2^2$ = const. are circles and fixing $Y_1 = +1$ determines only Y_1 uniquely without restricting the sign of Y_2.

[13] If c is not real then the argument becomes technically more complicated but in essence it remains the same, i.e., one still has to fix the reflection ambiguity.

(62). Of course, both methods yield the same result:

$$Y_1 = Y_1^h \,, \tag{74}$$

$$Y_2 = -\frac{1}{d}\tan\left(fx^1\right), \tag{75}$$

$$\Phi = \frac{2}{a}\ln\left(\cos\left(fx^1\right)\right). \tag{76}$$

with

$$d := \frac{a}{f}\,, \quad f := \sqrt{a(a(Y_1^h)^2 + 4|c|^2 b)}\,. \tag{77}$$

It has been assumed that $a, b > 0$ and thus $\mathcal{C}$ has to be positive. Integration constants have been absorbed by shifts of x^1 and ϕ. The sign of Y_2 has been chosen such that $\Phi \to -\infty$ in the "asymptotic region" $fx^1 = \pi/2$. In accordance with the previous discussion the asymptotic point $\Phi = -\infty$ may be added which yields the topology of a sphere. Note that for negative a the quantity Y_2 has to be positive; this change of sign is in accordance with the first remark in 4.1 because a change of the sign of Φ also changes the sign of Y_2 according to (62).

5.4 Local quantum triviality versus correlators

Local quantum triviality can be derived by looking at the effective action in terms of mean fields. It can be checked easily that the effective action, up to boundary contributions, is equivalent to the classical action in the gauge (53). We denote

$$\langle\omega_2\rangle = \frac{\delta \ln \mathcal{Z}(j,J)}{\delta j_\omega} \tag{78}$$

by ω_2, where $\mathcal{Z}$ is given by (68), and similarly for all other variables. One obtains by virtue of (62)-(64) upon substitution of the sources into $\int d^2x(j_\omega\omega_2 + j_i e_2^i)$ (the other terms cancel trivially or yield boundary contributions)[14] the effective Lagrangian density

$$L_{\text{eff.}} = -(\partial_1\Phi - Y_2)\omega_2 - \partial_1 Y_1 e_2^1 - (\partial_1 Y_2 + (Y_1^2 + Y_2^2)U(\Phi) + 2|c|^2 V(\Phi))e_2^2\,. \tag{79}$$

Recalling the redefinitions (61) and corresponding ones for the Zweibeine ($e_2^1 := (ce_2^* + c^*e_2)/(2|c|^2)$, $e_2^2 := i(ce_2^* - c^*e_2)/(2|c|^2) = -(\det e)/(2|c|^2)$) it is seen that (79) is nothing else than the classical action in the gauge (53); to this end it is helpful to rewrite (79) (up to boundary terms) as

$$L_{\text{eff.}} = Y_1\partial_1 e_2^1 + Y_2(\partial_1 e_2^2 + \omega_2) + \Phi\partial_1\omega_2 + \det e((Y_1^2 + Y_2^2)U(\Phi)/(2|c|^2) + V(\Phi))\,. \tag{80}$$

[14]The explicit form of L_{amb} mentioned above is neither relevant for the discussion of local quantum triviality nor for the special correlators considered below and thus we do not include it here.

With (61) this reproduces exactly (57). Therefore, as expected from the Minkowskian case the theory is locally quantum trivial.

Local quantum triviality by no means implies that all correlators are trivial [55]. For instance[15], the Lorentz invariant 1-forms $Y^a(x)e_a(x)$ and $\epsilon_{ab}Y^a(x)e^b(x)$ exhibit interesting behaviour if delocalised

$$\langle Y^a(x)e_a(y)\rangle = \langle Y^a(x)\rangle\langle e_a(y)\rangle + \delta_1(x,y)\,, \tag{81}$$

$$\langle \epsilon_{ab}Y^a(x)e^b(y)\rangle = \epsilon_{ab}\langle Y^a(x)\rangle\langle e^b(y)\rangle + \delta_2(x,y)\,, \tag{82}$$

i.e. the quantities δ_1, δ_2 will be shown to be nonvanishing, while the remaining terms just yield the classical result expected from local quantum triviality discussed above. It is recalled that the state in (81) and (82) is fixed by choosing the integration constant $\mathcal{C}_0$. One may wonder why we take specifically the correlators (81), (82). The reason for this is twofold: first, they are Lorentz invariant locally in the coincidence limit and still retain global Lorentz invariance even non-locally.[16] Second, correlators which contain arbitrary powers of Φ, Y_1, Y_2 alone cluster decompose into products of their classical values, so one needs at least one gauge field insertion. So the correlators above are the simplest non-trivial examples of non-local correlators which retain at least global Lorentz invariance. Finally, it should be mentioned that the objects

$$\delta_1(x,y) = \frac{\delta}{\delta j_1(y)}Y_1(x) + \frac{\delta}{\delta j_2(y)}Y_2(x)\,, \tag{83}$$

$$\delta_2(x,y) = \frac{\delta}{\delta j_2(y)}Y_1(x) - \frac{\delta}{\delta j_1(y)}Y_2(x) \tag{84}$$

are 1-forms and thus naturally may be integrated along a path.

To determine δ_1 and δ_2 one needs the variations $\delta Y_i(x)/\delta j_j(y)$ with $i,j=1,2$ and Y being the solution (65) and (66), resp. The state is defined so that (70) holds, thus fixing the integration constant in (66). For $i=1$ the expressions follow straightforwardly from (65):

$$\frac{\delta}{\delta j_1(y)}Y_1(x) = \int_{\bar{x}^1}^{x^1} dz^1 \delta^2(y-z) \qquad\qquad \frac{\delta}{\delta j_2(y)}Y_1(x) = 0 \tag{85}$$

The lower integration limit should be chosen conveniently. To fix the remaining free parameters induced in this way it is natural to require a vanishing result in

[15]Expectation values involving the e_a and ω alone would require the inclusion of L_{amb}.

[16]By insertion of a Wilson line $\exp i\int_x^y \omega^a{}_b$ integrated over a certain path starting at the point x and ending at the point y, one can construct correlators similar to the ones in (81), (82) which are then locally Lorentz invariant. However, they do not allow an immediate physical interpretation as known from other topological models. Since the main message we want to convey is that the quantities $\delta_{1,2}$ are non-vanishing we only present the simpler calculation of the correlators (81), (82).

the coincidence limit $x = y$. Following the choice of [55] we define $\theta(0) = 1/2$ and choose $\bar{x}^1 = y^1$. More involved is the variation of Y_2 as—beside the explicit appearance of j_i—the dilaton implicitly depends on the sources. Therefore the variation yields an integral equation, in particular for j_2

$$\begin{aligned} \frac{\delta}{\delta j_2(y)} Y_2(x)\Big|_{j=0} = \frac{1}{2Y_2} \Bigg(& 2e^{-Q} \partial_1^{-1}{}_{xz} \big(e^Q \partial_1 \Phi \delta^2(y-z) \big) \\ & + \Big(\frac{\partial}{\partial \Phi} Y_2^2 \Big) \int^{x^1} dz \frac{\delta}{\delta j_2(y)} Y_2(z) \Bigg) \Bigg|_{j=0} . \end{aligned} \tag{86}$$

Its solution can be obtained straightforwardly as

$$\begin{aligned} \frac{\delta}{\delta j_2(y)} Y_2(x) = [e^Q Y_2]_y \Big(\big[\frac{e^{-Q}}{Y_2} \big]_x + (\partial_{x^1} Y_2) \big(F(x) - F(y) \big) \Big) & \\ \big(\theta(x^1 - y^1) - \frac{1}{2} \big) \delta(x^2 - y^2) \, , & \end{aligned} \tag{87}$$

where $F(x)$ is the abbreviation for

$$F(x) = \int^{x^1} dz^1 \frac{e^{-Q}}{Y_2^2} \ . \tag{88}$$

For sake of completeness it should be noted that a similar calculation can be done for the variation with respect to j_1. This correlator depends on an additional integral

$$G(x) = \int^{x^1} dz^1 Y_2^{-2} \ . \tag{89}$$

Using the same prescription as the one that led to (87) the result becomes

$$\begin{aligned} \frac{\delta}{\delta j_1(y)} Y_2(x) = Y_1^h \Big\{ & \frac{1}{Y_2(x)} \big(e^{Q(y)-Q(x)} - 1 \big) + (\partial_{x^1} Y_2) \\ & \Big(e^{Q(y)} \big(F(x) - F(y) \big) - G(x) + G(y) \Big) \Big\} \big(\theta(x^1 - y^1) - \frac{1}{2} \big) \delta(x^2 - y^2) \ . \end{aligned} \tag{90}$$

By a dilaton-dependent conformal transformation $e_\mu^a \to \tilde{e}_\mu^a = e^{\rho(X)} e_\mu^a$ one can map one dilaton gravity model upon another. The potentials $\tilde{U}$ and $\tilde{V}$ will be, of course, different from the original ones U and V (cf. e.g. Eq. (3.41) in ref. [1]). This transformation is usually called the transition to another conformal frame, although models corresponding to different frames in general are already inequivalent at the classical level (because of possible singularities in the transformation). By a suitable choice of $\rho(X)$ one can achieve $\tilde{U} = 0$ (and, consequently, $\tilde{Q} = 0$). Note that in a conformal frame with $Q = 0$ Eq. (90) vanishes

identically. Thus, the quantity $\delta_2(x, y)$ captures the dependence on the conformal frame and vanishes if there is no kinetic term for the dilaton. In contrast, (87) is non-vanishing even for this simple case and (85) yields a frame independent contribution. Thus, $\delta_1(x, y)$ encodes both, frame independent as discussed in [55] as well as frame dependent information.[17]

6 Conclusions

The methods which have been very successful in two-dimensional gravity [1] are extended to Euclidean signature models. To establish classical and quantum integrability of dilaton gravities also in Euclidean space, is a necessary prerequisite to treat strings and, in particular, Liouville theory. We are able to explicitly construct all local classical solutions for all theories of this type. In quantum theory we perform the path integral nonperturbatively, where, however, problems specific for Euclidean signature have to be overcome: the Eddington-Finkelstein gauge which plays a key role in the comparatively simpler case of Minkowski space theories has no counterpart here, so that a new type of complex gauge fixing had to be introduced. This again permitted us to follow in broad lines the strategy, well-tested in the Minkowski space. As in that case we observe *local* quantum triviality. Despite this property, non-local correlator functions exist. In examples we show how to compute them explicitly in our approach. We, therefore, believe to have performed a very important first step towards the study of general correlators for the Liouville model which may be related to matrix models and strings. This will be the subject of a future publication.

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[17]At this point we should clarify a misleading statement in ref. [55]: while everything is correct until Eq. (4.37), Eq. (4.38) and subsequent Eqs. depending on it are only true for $Q = 0$. In particular, the correlator Eq. (4.42) in addition to the purely topological information discussed in that paper encodes also information about the conformal frame.

A Equivalence of first and second order formulations

The proof of classical equivalence between first and second order formulations differs in details only from the corresponding one for Minkowski signature [1,56]. Quantum equivalence has been demonstrated in [49].

The action (2) is rewritten in the component form,

$$L = \frac{1}{2}\int_{\mathcal{M}} d^2\sigma \left[Y^a (D_\mu e^a_\nu)\tilde{\varepsilon}^{\mu\nu} + \Phi \partial_\mu \omega_\nu \tilde{\varepsilon}^{\mu\nu} + \boldsymbol{e}\mathcal{V}(Y^2, \Phi) \right] , \tag{91}$$

where

$$\boldsymbol{e} = \det e^a_\mu = -\frac{1}{2}\varepsilon^{ab}\tilde{\varepsilon}^{\mu\nu} e^a_\mu e^b_\nu \,. \tag{92}$$

The two Levi-Civita symbols, ε^{ab} and $\tilde{\varepsilon}^{\mu\nu}$, are defined as $\varepsilon^{12} = \tilde{\varepsilon}^{12} = 1$. The one with anholonomic indices (ε^{ab}) is tensorial, while the symbol with holonomic indices ($\tilde{\varepsilon}^{\mu\nu}$) is a tensor density.

The connection ω_μ is split into the Levi-Civita part $\hat{\omega}$ and the torsion part t_μ, $\omega_\mu = \hat{\omega}_\mu + t_\mu$. By definition, the Levi-Civita connection

$$\hat{\omega}_\mu = \boldsymbol{e}^{-1} e^a_\mu \tilde{\varepsilon}^{\rho\sigma}(\partial_\rho e^a_\sigma) \tag{93}$$

corresponds to vanishing torsion:

$$\tilde{\varepsilon}^{\mu\nu}\left(\partial_\mu e^a_\nu + \varepsilon^{ab}\hat{\omega}_\mu e^b_\nu\right) = 0 \tag{94}$$

For $\mathcal{V}(Y^2, \Phi)$ given by (4) the equation of motion for Y^a reads

$$t_\mu \tilde{\varepsilon}^{\mu\rho} + \boldsymbol{e} U(\Phi) Y^a \varepsilon^{ac} e^\rho_c = 0 \,, \tag{95}$$

where e^ρ_a are the inverse components of the ones in the Zweibein 1-form. This equation is linear and, therefore, it may be substituted back into the action to eliminate t:

$$L = \frac{1}{2}\int_{\mathcal{M}} d^2\sigma \left[\Phi \partial_\mu \hat{\omega}_\nu \tilde{\varepsilon}^{\mu\nu} + \boldsymbol{e} U(\Phi)\left((\partial_\rho \Phi) e^\rho_c \varepsilon^{ac} Y^a - \frac{1}{2}Y^2 \right) + \boldsymbol{e} V(\Phi) \right] . \tag{96}$$

The equation of motion for Y^a following from this action is again linear:

$$Y^a = (\partial_\rho \Phi) e^\rho_c \varepsilon^{ac} \,. \tag{97}$$

We also note that

$$\partial_\mu \hat{\omega}_\nu \tilde{\varepsilon}^{\mu\nu} = -\boldsymbol{e}\frac{R}{2} \,, \tag{98}$$

where R is the curvature scalar. With our sign conventions $R = 2$ on unit S^2. Next we substitute (97) and (98) in (96). We also note that we can re-express everything in terms of the Riemannian metric $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$ instead of e^a_μ. As a result, we obtain the second order action (1) (up to an irrelevant overall factor of 1/2).

B Compact manifolds

On a compact connected manifold $\mathcal{M}$ the Laplace operator Δ has a zero mode. Consequently, the conformal anomaly (36) has to be modified, as well as the local and non-local actions (38) and (37). This procedure was considered in detail by Dowker [57]. One has to subtract the zero mode contribution from (36) so that the modified expression for the anomaly reads:

$$T_\mu^\mu = \frac{N-1}{24\pi}[R+\tilde{\mu}] - \frac{N-1}{\mathcal{A}}\,, \tag{99}$$

where $\mathcal{A}$ is the area of $\mathcal{M}$.

The non-local action (37) receives several additional terms [57], and the Liouville action (38) should contain a non-local term $\frac{N-1}{2}\ln\mathcal{A}$. Accordingly, one has to add to the first-order action (2) the term:

$$L_{\text{area}} = \beta\ln\mathcal{A}\,, \tag{100}$$

where the constant β can be fixed by the considerations presented above, but we consider here general values of β. A more convenient representation uses an auxiliary variable h:

$$\tilde{L}_{\text{area}} = \beta(h\mathcal{A} - 1 - \ln h)\,. \tag{101}$$

The action (38) is the only one directly derived from strings. It corresponds to $\alpha = 0$ in the potentials (41). At the classical level, for that value of α adding the term (101) results in shifting b to

$$\hat{b} = b + \beta h\,. \tag{102}$$

Therefore, all solutions obtained in sec. 4 remain valid after the replacement $b \to \hat{b}$. The exact value of this shift is fixed by a "global" equation of motion following from (101):

$$\mathcal{A} = 1/h\,. \tag{103}$$

For the present model (cf. Case sI of sec. 4), we have

$$\mathcal{A} = \theta_0 \int_{-\infty}^{\Phi_0} e^{a\Phi} d\Phi = \frac{2\pi a}{\hat{b}}\,. \tag{104}$$

We recall that for infinite volume solutions no modification of the action is needed. Equations (103) and (104) yield

$$h = \frac{b}{2\pi a - \beta}\,. \tag{105}$$

Another problem arising for compact manifolds is that the equation of motion for Φ from the action (38),

$$R = -2\Delta\Phi\,, \tag{106}$$

has no smooth solution except for the case of vanishing Euler characteristic. In our approach this problem is resolved automatically, because the classical solution obtained in Case sI becomes compact after adding a point with $\Phi = -\infty$. This solution is even unique as we identify Φ with a coordinate on $\mathcal{M}$. Of course, a similar problem exists also in quantum theory where it should be solved by choosing appropriate boundary conditions.

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